10. Numerical Integration

10.1. Introduction

Sometimes a problem in Science or Engineering allows us to find an elegant solution that represents a simple, exact answer. Mathematics tells us that the area of a circle is exactly πr^2. We know that the distance travelled by a uniformly accelerating body is $v_0 t + \frac{1}{2}at^2$.

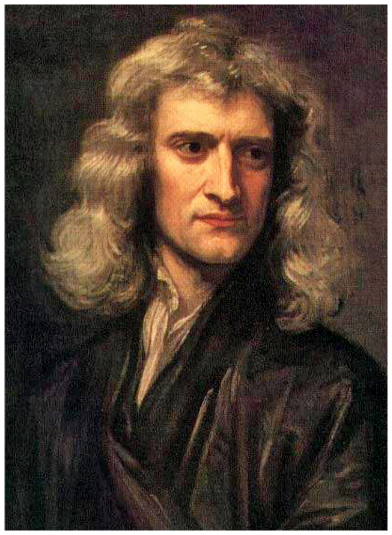
Elegant answers aren't always available, though. Often, a simple mathematical solution eludes us. This can happen because of some inherent feature of the problem that makes it mathematically difficult, or because the sheer size of the problem makes it intractible, or because we only have a little bit of data.

For problems like this, we need to apply brute force. We chip away at the problem, hoping to find an approximation that's good enough to satisfy our immediate needs. Fortunately, we often find that, by working hard enough, we can make our approximation as close to the true answer as we like.

One place this kind of problem crops up is in the evaluation of integrals. As you know if you've taken Calculus, the evaluation of integrals can be difficult. Much of what you learn in Calculus class consists of tricks for evaluating various kinds of integrals.

An integral is conceptually simple, though: it's just adding things up. Mathematician Gottfried Leibniz introduced the integral sign, $\int$, which is a stretched-out "S", for "Sum".

Since computers can add things very quickly and accurately, you'd think they'd be good at integration. In this chapter, we'll look at a couple of ways computers can help you deal with tricksy integralses. Techniques like this are called "numerical integration", since they compute the approximate values of definite integrals by using numbers, instead of finding an exact, symbolic, value.


Isaac Newton in 1689.
Source: Wikimedia Commons

Calculus (first called the "calculus of infinitesimals") was co-invented in the 17th Century by Gottfried Wilhelm von Leibniz in Germany and Isaac Newton in England. The two argued bitterly over which of them deserved credit. The Royal Society of London formed a committee chaired by Newton to investigate the dispute, and its report (written by Newton) ruled, unsurprisingly, in favor of Newton. Today mathematicians give both men equal credit.

Gottfried Wilhelm von Leibniz, *circa* 1700.
Source: Wikimedia Commons

10.2. Integrals

Imagine that you own a power company[1]. Being a good businessman, you keep a close eye on how much power your customers are using. Over the last year, you've observed that there's a lot of variation in power consumption over the course of a day. Figure 10.1 shows some of the data you've collected.

[1] Maybe you do. How would I know?

Kernkraftwerk Beznau, a nuclear power plant in Aargau, Switzerland. As a graduate student, I used to take a shortcut through the plant's parking lot every day on my way to and from work. I wonder how the guards there today would react to a scruffy, backpack-laden student walking by in the night. Would they wave and say *"guten abend!"* as they did to me?

Source: *Wikimedia Commons*

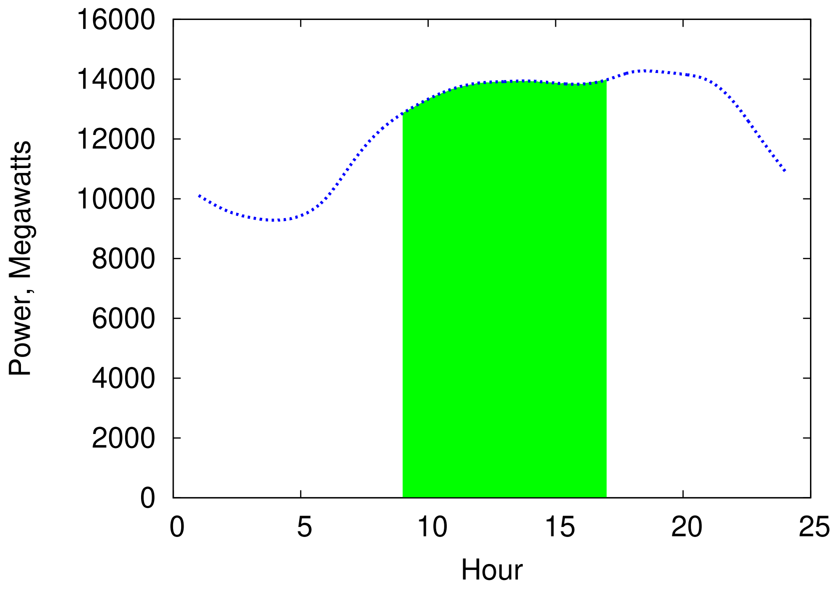


Figure 10.1: Average hourly power consumption on a New England power grid. The shaded region shows times between 9 am and 5 pm.

Source: *eia.gov*

In the early morning, people are asleep and power usage is low. Usage picks up during the day, falls off a little during commuting time, then surges at night as people turn on their lights, TVs, ovens, and popcorn poppers.

Figure 10.1 shows power usage on the vertical axis and hour of the day on the horizontal axis. Power is the rate at which energy is flowing, and it's measured in watts (or megawatts in this case). This *power* data is interesting, but if we want to know how much coal or gas or sunlight our energy company needs, we have to know how much *energy* people are consuming. How can we determine that? To figure that out, let's start with a simpler example.

A microwave oven might draw 1 kilowatt of power while it's running. If the microwave runs for half an hour, the amount of energy it uses is:

$$1 \text{ kilowatt} \times \frac{1}{2} \text{ hour} = 0.5 \text{ kilowatt-hours}$$

Kilowatt-hours is a unit of energy. When we do the calculation above, it's equivalent to determining the shaded area shown in Figure 10.2. Mathematically, we could express it like this:

$$E = P(t)\Delta t$$

where E is energy, P(t) = 1000 watts (a steady, unvarying power consumption), and Δt is the amount of time the microwave oven is running.

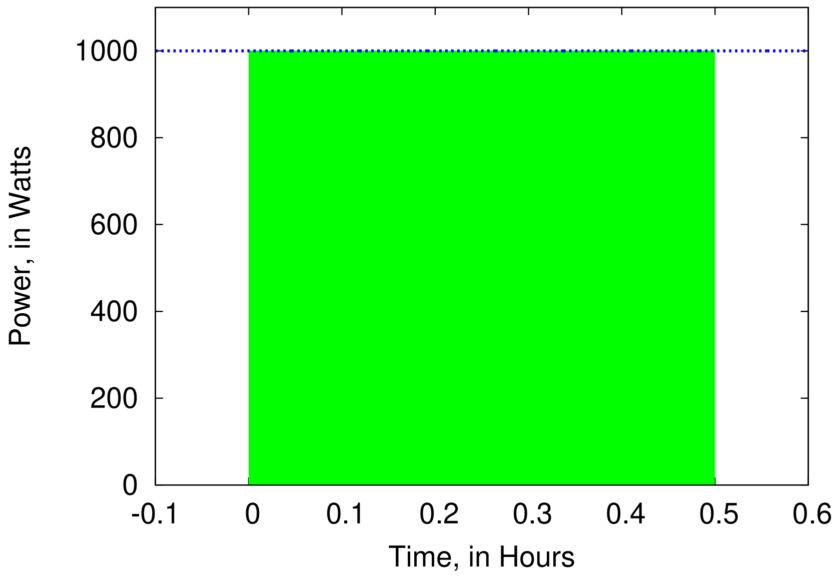
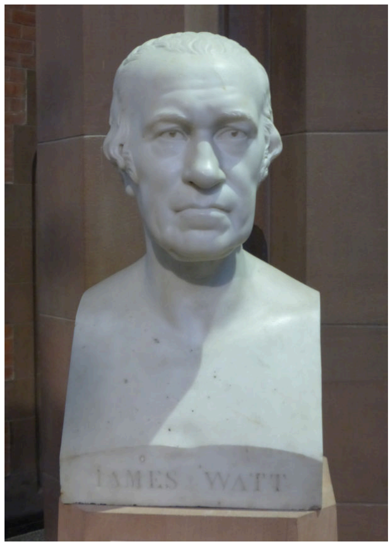


Figure 10.2: If we assume that the microwave oven uses a constant amount of power while it's running, the energy it consumes is equal to the shaded area in this graph, which is the power (P, in watts) multiplied by the amount of time (Δt, in hours).

Unfortunately for our power company, Figure 10.1 shows that our customers don't use the same amount of power all the time, so we can't just do a simple multiplication to find out how much energy they use. If we want to know how much energy is used between 9 am and 5pm, we need to determine the size of the shaded area in Figure 10.1. Mathematically, that's equivalent to evaluating the following integral equation:

$$E = \int_{9am}^{5pm} P(t)dt$$

The important thing to remember is that the integral is just the area under the curve defined by the P(t) function. As we saw in the microwave oven example, this is sometimes trivial to calculate. In calculus class we learn some mathematical tricks to evaluate the integrals of more complicated functions.



Scottish inventor and entrepreneur James Watt, most famous for the invention of the steam engine, for whom the unit of power is named.

Source: Wikimedia Commons

In the following, we'll see how to use a computer program to estimate the value of such integrals without using any tricks. To do so, we'll combine a little modern computing power with the basic principles that Newton and Leibniz used when they invented calculus way back in the 1600s.

The integral here is called a "definite integral" because it only covers a range between two limits (9 am and 5 pm in this case). The techniques we'll talk about in this chapter are all for finding approximate values for definite integrals.

10.3. Slicing up the Problem

If Newton and Leibniz had been familiar with microwave ovens and power plants, they'd have approached the problem this way: slice up the area into manageable bits.

We can slice a function up so that each slice is approximately rectangular.
Source: *Wikimedia Commons*

The inventors of calculus realized that the area under a curve could be approximated by the total area of a row of rectangles, as in Figure 10.3, like slices from a loaf of bread. As in the microwave oven example, the area of each rectangle is easy to calculate.

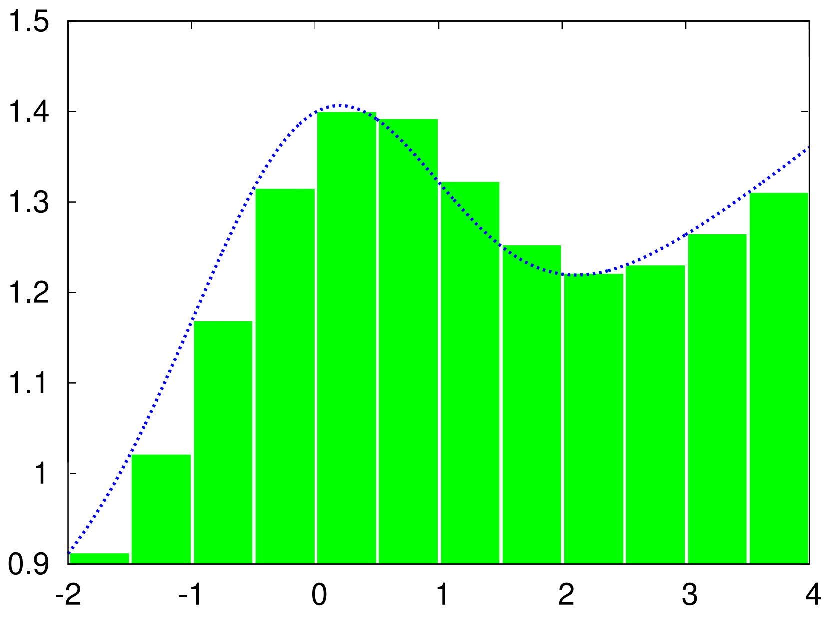


Figure 10.3: The area under the smooth curve is approximately the same as the total area of a row of rectangles of various heights.

If we want a better approximation, we can just use thinner slices, as in Figure 10.4. Mathematicians have found that in some cases we can arrive at the exact area under the curve by mathematically determining what the area *would* be if the width of the slices went to zero. Sometimes we can't calculate this limit, though. In those cases, we have to be satisfied with an approximation, but that's not so bad, because we can often make our approximation as accurate as we want by choosing the width of our slices.

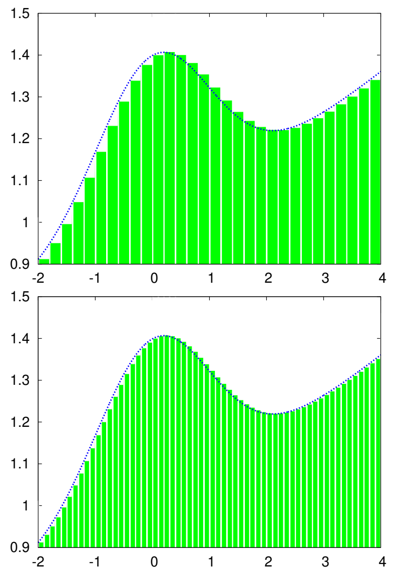


Figure 10.4: More slices give a better approximation.

Now that you have some experience writing programs, you can probably already see how we might do this with a computer. If we wanted to add up the areas of the rectangles shown in Figure 10.5 we could use a "`for`" loop. It might look something like this:

```
deltax = (xmax-xmin)/nslice;
x = xmin;
for ( i=0; i<nslice; i++ ) {
  height = f(x);
  area += deltax * height;
  x += deltax;
}
```

where `nslice` is the number of slices, and `deltax` is the width of each slice. There are several different ways we could make the slices, but for now let's put the upper left-hand corner of each rectangle so that it just touches the curve we're trying to approximate. The function `f(x)` gives the height of the curve at each value of `x`. The height of the first rectangle is `f(xmin)`, so its area is `f(xmin)` times `deltax`. Each time around the loop we add the area of the current rectangle to to the total area, then move right by a distance `deltax` and do it again, until we get to the last rectangle.

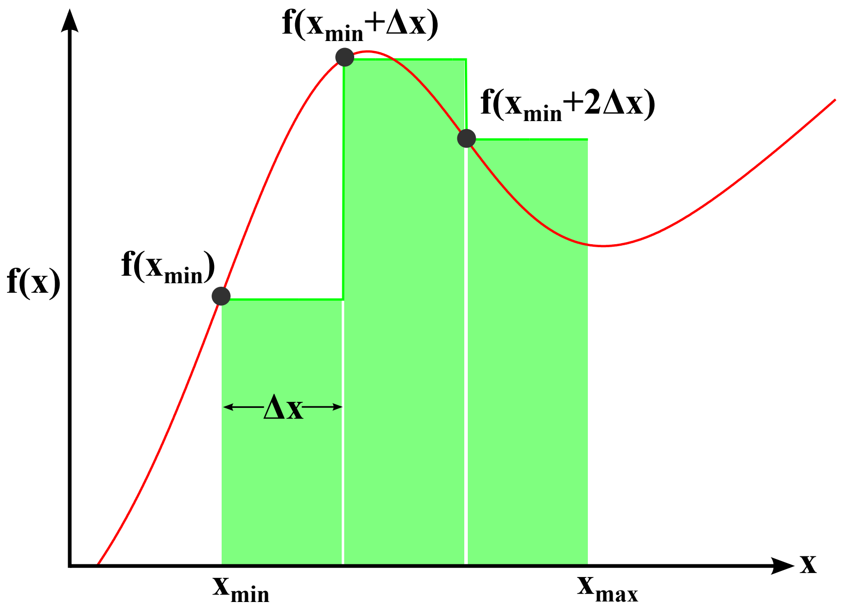


Figure 10.5: Approximating the area under the curve, between `xmin` and `xmax`, with three rectangles of width Δx.

Let's apply this technique to our power company. Imagine that we've asked our minions to give us a data file containing two columns of data: The hour of the day (in 24-hour time) and the average amount of power used at that hour (in megawatts). The file might look like Figure 10.6.

Program 10.1 is designed to read this file and estimate the area under the power curve between 9 am and 5 pm. Notice that we've placed the slices as shown in Figure 10.5, which means that each rectangle starts on the hour and covers the time until the next hour begins. This means that we don't want to include the 5:00 measurement (hour 17) in the total area, since we're assuming that this measurement is an estimate of the power used between 5 pm and 6 pm, which is outside the range we're interested in. (See Section 10.5 for more about this.)

Figure 10.6: The contents of the file "`power.dat`", containing hourly power measurements from our power company:

```
 1   10110.66
 2    9636.32
 3    9376.79
 4    9283.72
 5    9433.38
 6   10025.68
 7   11181.30
 8   12193.26
 9   12855.83
10   13332.67
11   13685.37
12   13871.35
13   13918.52
14   13935.75
15   13867.86
16   13833.21
17   13976.59
18   14238.24
19   14253.39
20   14163.90
21   13948.21
22   13220.34
23   12059.76
24   10920.85
```

Also notice that, even though each slice has a width of 1 hour, the program goes ahead and explicitly multiplies by this width, just to make it clear that we're calculating the area of the rectangular slice by multiplying its height times its width. If our data were at half-hour intervals, we'd change the 1.0 to 0.5.

Program 10.1: power.cpp

```
#include <stdio.h>
int main () {
  int hour[24];
  double power[24];
  FILE *input;
  int i;
  double area=0.0;

  input = fopen("power.dat","r");
  for ( i=0; i<24; i++ ) {
    fscanf( input, "%d", &hour[i] );
    fscanf( input, "%lf", &power[i] );
  }
  fclose ( input );

  for ( i=0; i<24; i++ ) {
    // Don't include the 5pm hour:
    if ( hour[i] >= 9 && hour[i] < 17 ) {
      area += power[i]  * 1.0; // 1 hour.
    }
  }
  printf ( "Total energy from 9am-5pm is %lf Mw-hours\n", area );
}
```

Exercise 51: Power to the People!

Using *nano*, create the file `power.dat` by entering the data from Figure 10.6. Create, compile and run Program 10.1. Make a note of the result, for later use.

By looking at Figure 10.1, make a rough estimate of the 9 to 5 power usage by multiplying the curve's approximate height, in megawatts, by 8 hours (the interval between 9 am and 5 pm). Is your rough estimate consistent with Program 10.1's estimate?

10.4. Trapezoids

The technique above would probably work fine if we used enough slices, but as you can see from Figure 10.5 the rectangles might not fit the curve very well if we only use a small number of slices, or if the curve goes up and down a lot over distances narrower than the width of a slice.

In the preceding section, we placed our rectangles on the right-hand side of the curve. Mathematicians call a this a "Right Riemann Sum". We could alternatively have chosen to put them on the left-hand side, as shown in the middle graph of Figure 10.7 (called a "Left Riemann Sum"). Neither of these rectangular sums fits the curve particularly well, but what if we averaged the two? That's what's shown in the bottom graph.

The slices in the last graph aren't rectangles. Instead, they're "trapezoids". A trapezoid is a four-sided figure with two or more parallel sides (unlike a rectangle which must have two pairs of parallel sides). We can use trapezoidal slices to estimate the area under the curve much more efficiently than with rectangular slices. These new slices are the same width as the rectangular ones, but their top edge has *both* corners, right and left, on the curve.

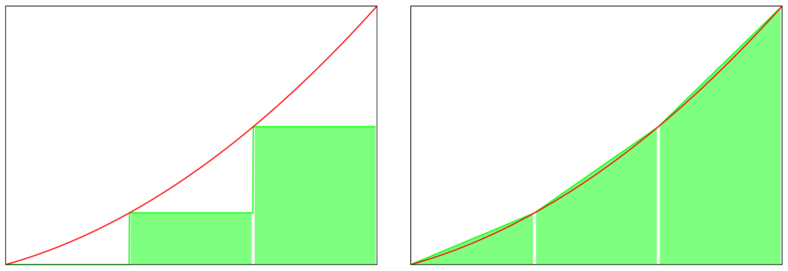
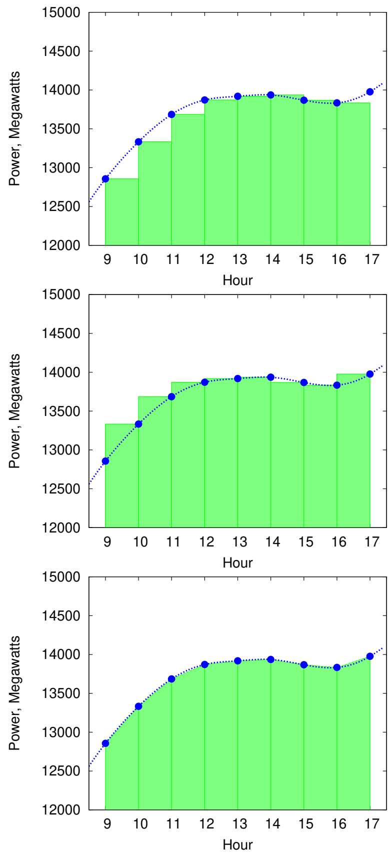


Figure 10.7:
Top: Right Riemann Sum
Middle: Left Riemann Sum
Bottom: Trapezoid Sum



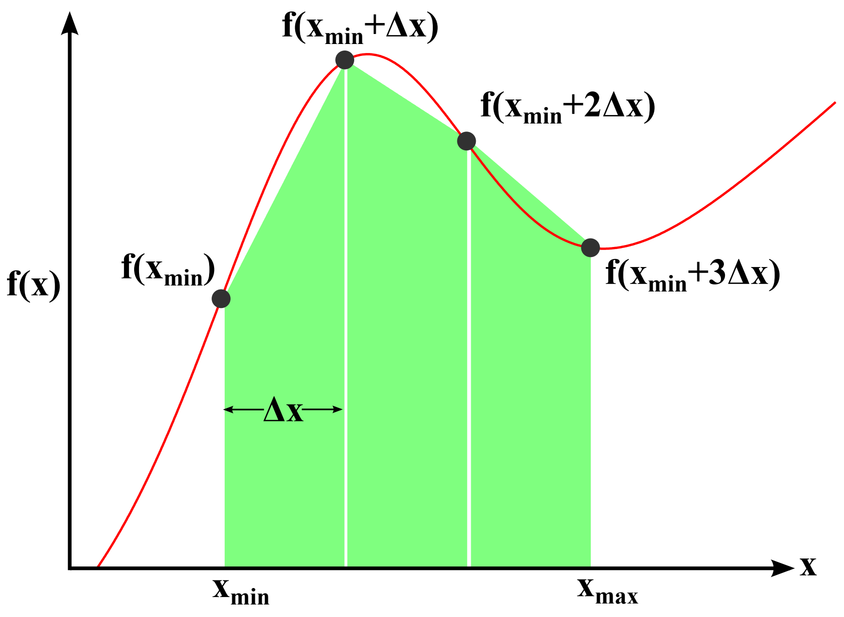


Figure 10.8: Approximating the area under the curve, between `xmin` and `xmax`, with three *trapezoids* of width Δx. Compare this with Figure 10.5, which uses rectangles.

Figure 10.8 shows how we might slice up an area into trapezoidal sections. As before, we can add up the areas of the slices to get an estimate of the area under the curve, but to do this we'll first need to know how to find the area of a trapezoid.

It turns out that this isn't so hard. The area of a rectangle is its height times its width, but the area of a trapezoid is its *average* height times its width. You can see why this is so by looking at Figure 10.9.

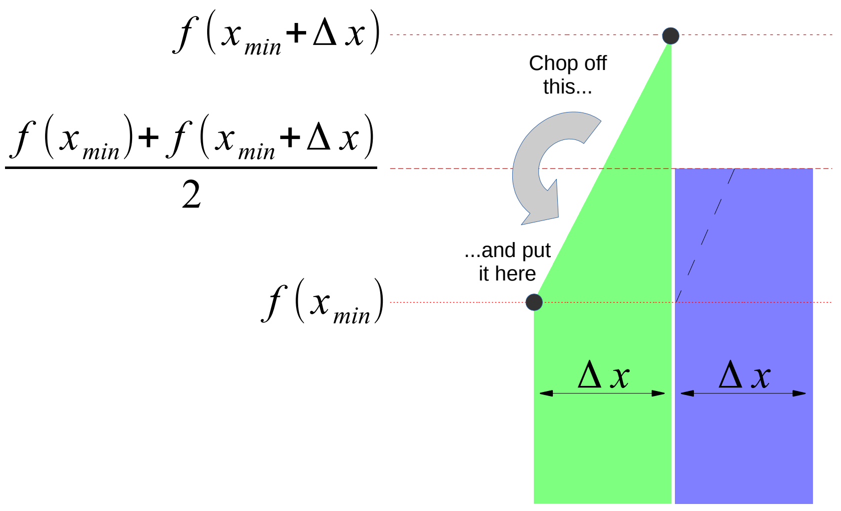


Figure 10.9: How to find the area of a trapezoidal slice.

Once we know how to find the area of a trapezoid, it's easy to modify Program 10.1 so that it uses this shape instead of rectangles. We only need to change one line. The result is Program 10.2

Program 10.2: power.cpp, with trapezoids instead of rectangles

```
#include <stdio.h>
int main () {
  int hour[24];
  double power[24];
  FILE *input;
  int i;
  double area=0.0;

  input = fopen("power.dat","r");
  for ( i=0; i<24; i++ ) {
    fscanf( input, "%d", &hour[i] );
    fscanf( input, "%lf", &power[i] );
  }
  fclose ( input );

  for ( i=0; i<24; i++ ) {
    // Don't include the 5pm hour:
    if ( hour[i] >= 9 && hour[i] < 17 ) {
      area += 0.5 * (power[i] + power[i+1]) * 1.0; // 1 hour.
    }
  }

  printf ( "Total energy from 9am-5pm is %lf Mw-hours\n", area );
}
```

The hammer dulcimer has a trapezoidal shape. The American folk band "Trapezoid" took its name from the shape of this instrument.
Source: *Wikimedia Commons*

Program 10.2 multiplies the width of each trapezoid by its average height. The average height is:

$$\frac{\texttt{power[i]} + \texttt{power[i+1]}}{2}$$

This way of approximating the value of an integral is called the "trapezoid rule".

Exercise 52: More Power to You!

Modify your `power.cpp` program so that it looks like Program 10.2. Compile and run it. How does the result compare with the result from the previous version? Does this agree with what you'd expect after looking at Figure 10.7?

10.5. Fencepost Problems

Consider the cheery scene below. It shows a section of fence with six panels. Notice that this requires *seven* fenceposts.

When we look at data we need to think about whether we're interested in the measurements themselves or the intervals between them. There will always be one more fencepost than the number of panels, so if we're interested in the intervals between measurements we need to be careful not to overcount.

Imagine that each of the fenceposts represents a measurement of our power plant's output at a particular time of day. Notice that Programs 10.1 and 10.2 are careful to stop *before* the 5pm measurement. These programs add up the energy produced during the intervals between measurements.

This subtlety occurs often in programming. It's called a "fencepost problem". Whenever you write a program that works its way through a number of measurements, always stop and think about whether you're interested in the measurments or the "in-betweens".

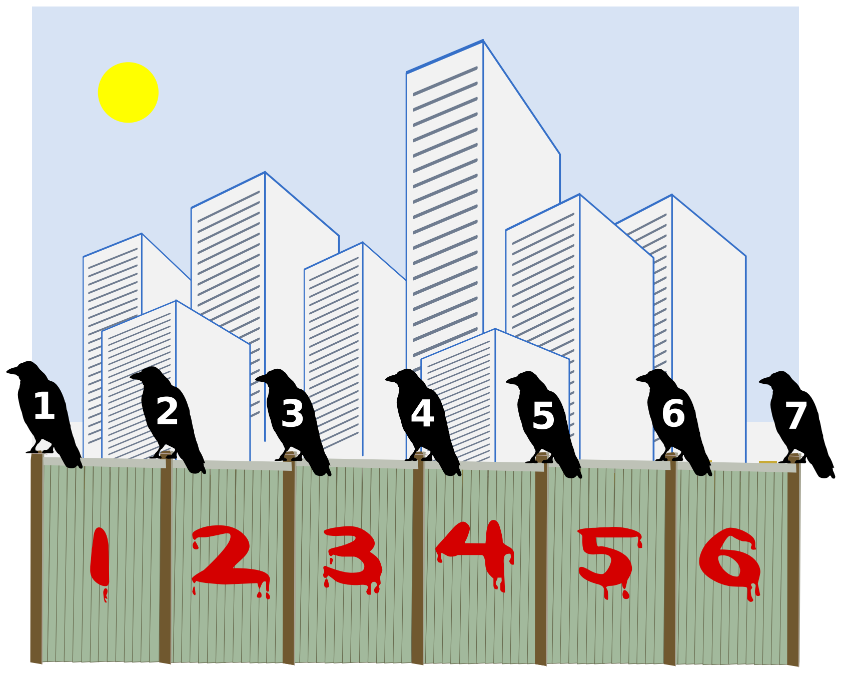


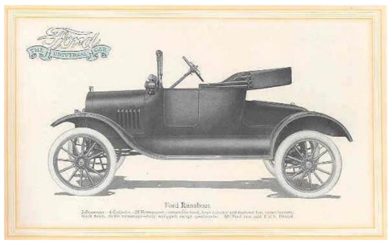
Figure 10.10: A fencepost problem. Are we interested in the number of panels or the number of crows?

10.6. Uneven Slices

In the preceding examples we've assumed that our measurements were evenly spaced. What if they're not, though? To explore that possibility, let's take a road trip!

In October 1925 Harry Lillis "Bing" Crosby and his pal Al Rinker set out from Spokane, Washington, bound for Los Angeles in Al's beat-up Ford Model T. Over the next three weeks they made their way down the coast, stopping whenever the car broke down and working a day or two to earn some money. The car survived until the outskirts of LA, where its engine finally died. They were taken in by Al's sister, the singer Mildred Bailey, who helped the boys find jobs performing in LA. Within a year they were hired by internationally-known bandleader Paul Whiteman. The rest, as they say, is history.

Source: Wikimedia Commons



Your car's odometer tells you how far you've driven, but modern cars don't measure distance directly. Instead, they use your velocity and a little bit of calculus to determine how many miles you've gone.

A mathematician would say that velocity is the time derivative of position, and she might write that relationship like this:

$$v = \frac{dx}{dt}$$

where dx represents a small change in position, dt is a small change in time, and v is the velocity. If the velocity doesn't change, then this is the same as saying that velocity is equal to distance divided by time. If we know the velocity and the time, we can calculate the distance as $distance = velocity \times time$.

If the velocity isn't constant, things get more complicated. In that case, our mathematician friend would tell us that we could find the distance like this:

$$\text{distance} = \int_{t_0}^{t_1} v(t)dt$$

where $v(t)$ is a function that tells us the velocity at a given time. The times t_0 and t_1 are when our trip starts and ends, repectively.

That looks complicated, but it's just like what we've been doing earlier in this chapter. If we're given some data about the car's velocity at various times, we can write a computer program to do the integral above and estimate the distance we've traveled.

Imagine we're going on a long trip. When we start, our velocity is zero. Then we pull out onto the highway and start driving. We can't drive at a constant speed, though. Sometimes traffic will slow us down. Sometimes we'll be driving on roads with higher or lower speed limits. Sometimes we'll be zoned out listening to our favorite tunes and find that we've spent the last ten miles drifting along behind a spluttering jalopy. Eventually, we'll reach our destination and our velocity will go back to zero again.

If we noted the time and our speed occasionally, the data might look like the large dots in Figure 10.11. We're lazy and easily distracted, so we haven't done the measurements at regular intervals.

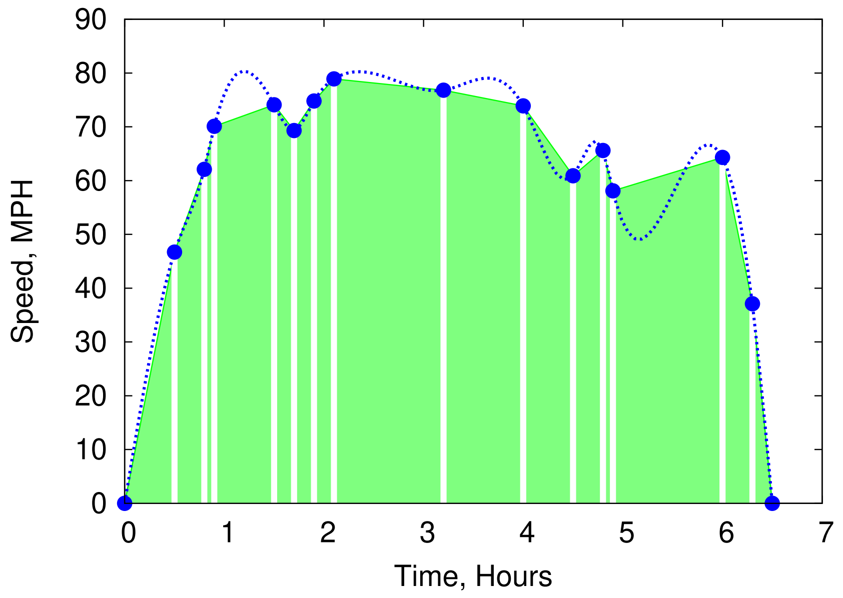


Figure 10.11: The large dots represent measurements of our speed. If we could have constantly monitored the speed it might have looked like the dashed curve. The whole trip takes about six and a half hours.

As you can see from the figure, we can draw trapezoids that connect the dots, just as we did with the power plant data. The only difference is that these trapezoids aren't all the same width. That's not a problem. We just need to take the width into account when we calculate the area of each trapezoid. Adding all of the areas together gives us the total area of the shaded region in Figure 10.11. The size of this shaded area is approximately equal to the distance we've traveled, as given by the integral equation our mathematician friend gave us above.

```
0.0 0.0
0.5 46.7
0.8 62.1
0.9 70.1
1.5 74.1
1.7 69.3
1.9 74.8
2.1 78.9
3.2 76.8
4.0 73.9
4.5 60.9
4.8 65.6
4.9 58.1
6.0 64.3
6.3 37.1
6.5 0.0
```

Figure 10.12: The file `roadtrip.dat`, used by Program 10.3. The first column is time, in hours. The second column is our speed, in miles per hour, at that time.

Program 10.3 does the work for us. It's similar to Program 10.2, but it doesn't assume that there's a particular number of measurements. The power plant program knew that there were 24 hours in a day, so it could assume there would be 24 measurements to read from its data file. On our road trip we just took some measurements at random times. On the next trip we might take more or fewer. Because of this, the new program doesn't start by reading all of the data into a fixed-size array. Instead, it uses a different strategy that just focuses on measurements one or two at a time, as they're read from the data file.

For each of the trapezoids in Figure 10.11 we need two pairs of time and speed measurements[2]. The difference in the two times tells us the width of the trapezoid. The average of the two speeds tells us the average height of the trapezoid. Program 10.3 waits until it has read the first two lines from the data file, then does these calculations to find the area of the first trapezoid. Then it continues on, doing the same for each subsequent pair of data points. Notice that, at any point in our

[2] See Section 10.5.

trip, the area we've accumulated so far will be equal to the distance we've traveled so far. To emphasize this, the program prints out the updated time and distance each time it reads a new data point from the file.

Program 10.3: roadtrip.cpp

```
#include <stdio.h>
int main () {
  double hour, velocity;
  double oldhour, oldvelocity;
  double height, width;
  double area=0.0;
  int nmeasurements=0;
  FILE *input;

  input = fopen("roadtrip.dat","r");
  while ( fscanf( input, "%lf %lf", &hour, &velocity ) != EOF ) {
    if ( nmeasurements != 0 ) {
      height = 0.5 * (oldvelocity+velocity);
      width = hour - oldhour;
      area +=  height*width;
      printf ( "Distance after %lf hours is %lf miles\n",
               hour, area );
    }
    oldhour = hour;
    oldvelocity = velocity;
    nmeasurements++;
  }
  fclose ( input );

  printf ( "Total distance is %lf miles\n", area );
}
```

Notice that the program uses two variables, `oldhour` and `oldvelocity`, to remember the previous data point's values. To make the program wait until there are at least two data points, we use the variable `nmeasurements` to count the number of points we've read so far, and test this value before we start doing any calculations. After the first trip around the loop, `oldhour` and `oldvelocity` have been set and we're ready to calculate the area of the first trapezoid during the next trip around the loop.

Remember that our result is only an estimate of the distance we've traveled. The estimate would be more accurate if we recorded more speed measurements during our trip. (If we'd only written down the speed at the beginning and end of the trip, the program would tell us that the distance was zero!)

10.7. Integrating Functions

In the exercises above, we've been finding the integral of a curve defined by data points, instead of being defined by some mathematical function. This is one common reason people use numerical integration: they have some data points, but don't know the underlying mathematics that generated them. You can't use calculus to compute the integral of a function if you don't know what that function is!

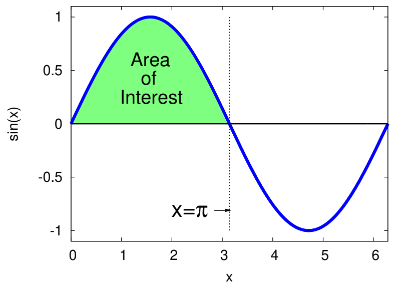


Figure 10.13: sin(x)

But what if you do know the function? As noted before, integration can be hard, and much of what we learn in calculus class is a set of tricks for finding the values of certain integrals. Sometimes, though, there are no applicable tricks that will let write down a value for a particular integral in terms of elementary functions (trigonometric functions, logarithms, *etc.*), or even "special" functions (the error function, the gamma function, *etc.*). Integrals of some seemingly simple expressions like $sin(sin(x))$ turn out to be impossible to evaluate.

Happily, in these cases the poor beleaguered mathematician can turn to numerical integration. As long as we can find the value of a function at any point within the range we're interested in, we can numerically approximate the value of the definite integral of the function over that range.

Let's look at an example where we *can* find the value of the integral mathematically. That will allow us to compare an exact mathematical solution to an approximate solution computed by the trapezoid rule.

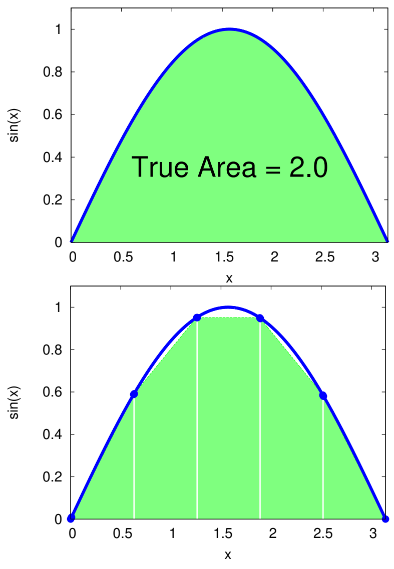


Figure 10.14: The integral of the sine function between 0 and π has a value of 2 (the area under the curve in the top graph). The bottom graph shows this area approximated by five trapezoidal slices.

For example, in calculus class we learned how to integrate the sine function (see Figure 10.13):

$$\int_A^B sin(x)dx = cos(A) - cos(B)$$

This tells us that the area under the sine curve between 0 and π is exactly $cos(0) - cos(\pi) = 2$, as shown in the top graph of Figure 10.14. Let's use the trapezoid rule to find this same area, and see how close it gets to the true answer.

That's what Program 10.4 does. Notice that, instead of using `sin(x)` directly, the program uses a function we define ourselves, named `func`. If we ever want to use this program to integrate something other than `sin(x)`, we'll only need to change this function definition. We can put anything we want to inside `func`. It could be as simple as `sin(x)` or as complicated as the "Red Baron" flight path we used in Chapter 9.

Program 10.4: integrate.cpp

```
#include <stdio.h>
#include <math.h>

double func( double x ) {
  double value;
  value = sin(x);
  return(value);
}

int main () {
  double x, delta, area=0;
  double height;
  double xmin=0.0, xmax=M_PI;
  int i, nsteps=5;

  delta = (xmax-xmin)/nsteps;
  x = xmin;
  for ( i=0; i<nsteps; i++ ) {
    height = ( func(x) + func(x+delta) ) / 2.0;
    area += delta * height;
    x += delta;
  }

  printf ( "Integral from %lf to %lf is %lf\n",
           xmin, xmax, area );
}
```

Slicing an onion in preparation for integrating it into dinner.
Source: Wikimedia Commons

The program specifies how many slices we want to use by setting the value of `nsteps`. (It's set to 5 here.) The variables `xmin` and `xmax` set the range over which we want to integrate. Notice that we use the symbol `M_PI` from `math.h` to set `xmax`, so we don't have to type out a long string of π's digits.

Most of the program's work is done in the "`for`" loop that works its way through the slices, one at a time, adding each area to the total area. As we go through the slices, we need to keep track of where we are on the x axis. Before we start the "`for`" loop, we set the value of `x` to `xmin`, and then we add the slice width (`delta`) to this when we're ready to go to the next slice.

To find the area of each slice, we multiply its average height by its width (`delta`). See Figures 10.8 and 10.9.

Exercise 53: Sines of the Times

1. Create, compile, and run Program 10.4. How close does its approximation come to the true value of the area?

2. Now change `nsteps` to 10, recompile, and run again. Is the answer closer to the true value? Try `nsteps` values of 100 and 1000. Does the program take noticeably longer if you use these large values for `nsteps`?

3. Now modify your program so that, instead of `sin(x)`, it finds the area under the curve `sin(sin(x))` between 0 and π. (See Figure 10.15.) Remember that it's not possible to find an exact value for this integral mathematically. Does the computer have any trouble finding an approximate solution? Write down this solution for later.

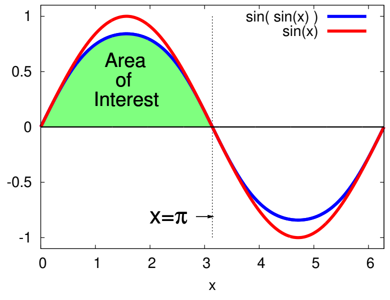


Figure 10.15: $sin(sin(x))$ and $sin(x)$

So far, this looks pretty good! We can come up with approximate values even for integrals that are impossible to solve exactly. Will this always work? Unfortunately, there are some pitfalls to look out for.

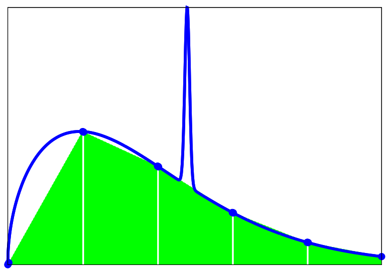


Figure 10.16: When integrating a function with sudden spikes or dips, we need to be careful to use a small enough slice width so that we avoid "stepping over" interesting features and missing them.

Figure 10.16 illustrates one potential problem. If the function we're integrating varies rapidly compared to the size of our slices, we may miss important features, possibly causing our estimate of the area to differ greatly from the true value.

Imagine what would happen if the spike in Figure 10.16 was very high. Even worse, what if the function goes to infinity at some values of x? Even mundane functions like $tan(x)$ or $1/x$ have infinities that we need to loook out for. The tangent of x goes to positive infinity as we approach $\pi/2$ from the right, and to negative infinity as we approach from the left!

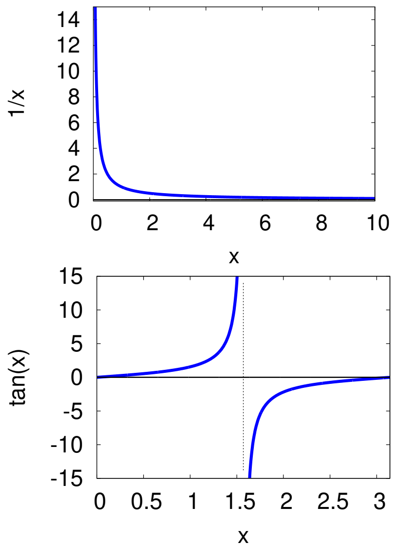


Figure 10.17: Functions like $1/x$ and $tan(x)$ go to infinity at some values of x. Our program would obviously have trouble if we included these x values in the range over which we tried to integrate.

If our program tried to find the value of the function at one of these x values, it would crash, but if we avoid these x values we may be missing a large part of the area we're trying to estimate. The area in the region around such points might even be infinite!

The trapezoid rule works well for a wide range of well-behaved functions, but it's important to be aware of the shape of the function you're integrating, and look out for problems like this. Even with ill-behaved functions, though, we can still use the trapezoid rule to find the integral over regions that don't include problematic points. We could, for example, integrate $tan(x)$ between 0 and 1 with no problem.

But what about...?

Wait a minute. Doesn't Program 10.4 calculate `func(x)` for the same values of `x` multiple times?

Take a look at Figure 10.8 again. If Program 10.4 were calculating the area of the first slice it would use `func(xmin)` for the height of the trapezoid's left-hand side and `func(xmin+delta)` for the right-hand side. For the second slice, it would use `func(xmin+delta)` for the left and `func(xmin+2*delta)` for the right. On the third slice the program would use `func(xmin+2*delta)` for the left and `func(xmin+3*delta)` for the right.

It takes the computer some time to process the statements inside a function. The program would run faster if we didn't have to calculate each of the middle `func` values twice. If `func` were more complicated, or if we had a lot of slices, the amount of time saved could be large.

We can do a little algebra and eliminate the duplication. If we have n slices we could write the sum of their areas like this:

$$\begin{aligned} A = \Delta x \frac{f(x_{min}) + f(x_{min} + \Delta x)}{2} &+ \\ \Delta x \frac{f(x_{min} + \Delta x) + f(x_{min} + 2\Delta x)}{2} &+ ... + \\ \Delta x \frac{f(x_{min} + (n-1)\Delta x) + f(x_{max})}{2} & \end{aligned}$$

Collecting terms, we can rewrite the equation like this, so that we only calculate the value of $f(x)$ once for each value of x:

$$A = \Delta x \left[\frac{f(x_{min}) + f(x_{max})}{2} + \sum_{i=1}^{n-1} f(x_{min} + i\Delta x) \right]$$

To take advantage of this, we could replace the loop in Program 10.4 with a loop like this:

```
 double sum = 0;
...
  for ( i=1; i<nsteps; i++ ) {
    sum += func(x+delta);
    x += delta;
  }
  area = delta*( (func(xmin) + func(xmax))/2.0 + sum );
```

There! We fixed it.

10.8. Negative Areas

Summer is here and it's time to fill the ol' backyard pool! Let's turn on the faucet. How long will it be before we can dive in? If we know how fast water is going into the pool, we should be able to calculate how much water has accumulated after a given amount of time. If the flow rate is given by a function $r(t)$, then the amount of water at time t is:

$$\text{amount of water} = \int_{t_0}^{t} r(t)dt$$

The function $r(t)$ might look like the blocky solid line in the top graph of Figure 10.18.

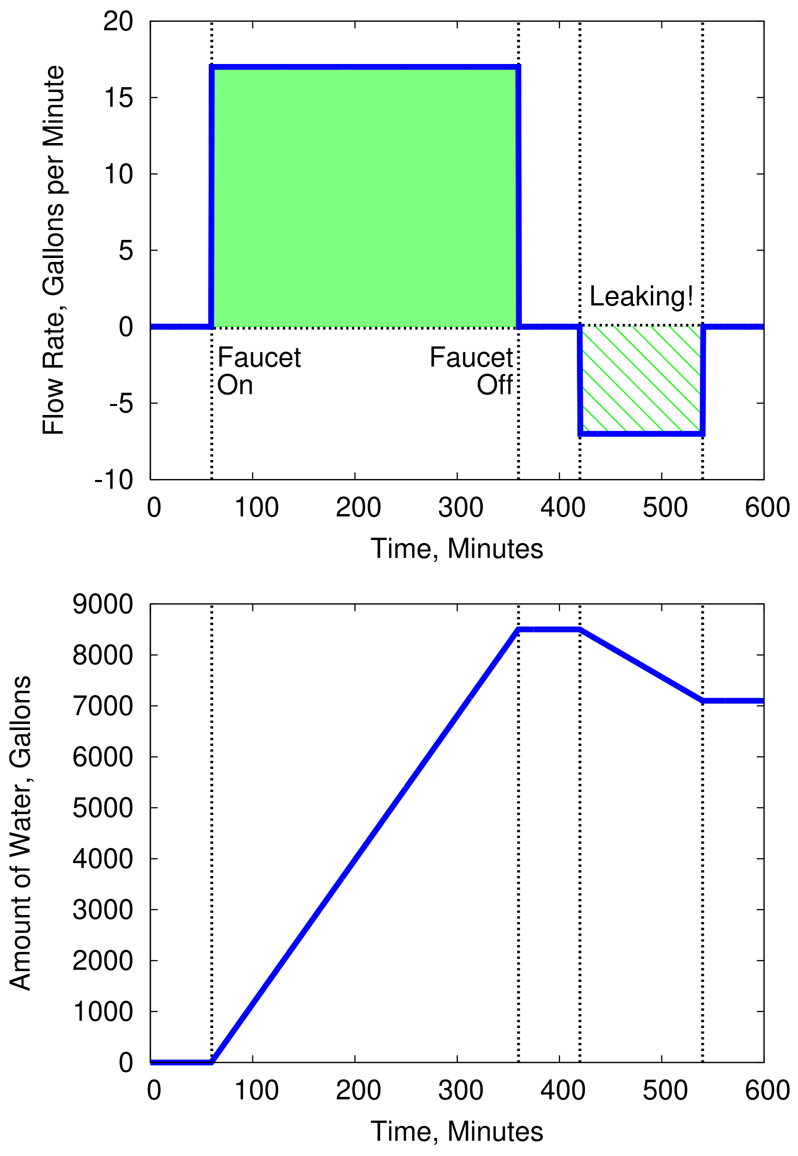


Figure 10.18: Filling a pool. The top graph shows the flow rate. Notice that a leak corresponds to a negative flow rate (water is flowing out of the pool instead of into it). The bottom graph shows how much water has accumulated in the pool Right before the leak the pool contained about 8,000 gallons of water, but the leak drained some of that off.

The amount of water in the pool is given by the integral of this curve, which is just the shaded area between the curve and the x axis in the top graph of Figure 10.18.

This figure shows the pool being filled for a while at a constant rate of 17 gallons per minute. Eventually, we accumulate about 8,000 gallons of water, as shown in the bottom graph, which shows the integral of $r(t)$ up to a given time.

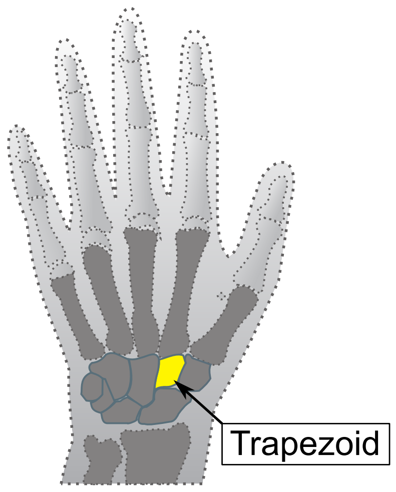
Uh oh! after filling the pool, it sprang a leak! This let water run out of the pool at a rate of 7 gallons per minute until we found the leak and patched it. If $r(t)$ is the rate of water *going into* the pool, that means that $r(t)$ had a *negative* value while the pool was leaking, as shown in the top graph of Figure 10.18.

If we used a computer program to slice up this curve and calculate its area, we'd find that the leak would contribute a negative amount to the sum because the height of the slices in this region would be negative. This is actually what we expect: The first part of the curve shows water flowing into the pool, and the second part of the curve shows water flowing out. To find out how much water we have at the end, we need to subtract the water that escaped through the leak.

The important thing to know is that we don't have to modify our programs in any way because of this. It just gets taken care of automatically when we multiply the height of a slice times its width. When the function we're integrating has a negative value, it contributes a negative amount to the total area.

10.9. A General-Purpose Trapezoid Integration Function

We can use the trick we learned in Section 9.17 of Chapter 9 to create a general-purpose function that can integrate anything using the trapezoid rule. Take a look at the function `trapint` defined below:



You carry a "trapezoid bone" in your wrist.

Source: Wikimedia Commons

```
double trapint ( double (*f)(double),
                 double xmin, double xmax,
                 int nsteps ) {
  double x, delta, area=0;
  double height;
  int i;

  delta = (xmax-xmin)/nsteps;
  x = xmin;
  for ( i=0; i<nsteps; i++ ) {
    height = ( f(x) + f(x+delta) ) / 2.0;
    area += delta * height;
    x += delta;
  }

  return( area );
}
```

"`trapint`" estimates the integral of the function "`f`" between `xmin` and `xmax` using the trapezoid rule. The argument `nsteps` specifies the number of trapezoids.

Notice that the first argument given to `trapint` is the name of the function we want to integrate. We could say `cos`, for example, to compute the area under some section of the cosine function, or `sqrt` to do the same for the square root function. But we aren't limited to the built-in functions. We also use `trapint` with a function we write ourselves. The only restriction is that the function we use must return a `double` value and take one `double` argument.

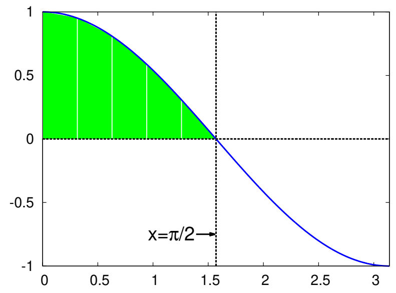


Figure 10.19: The integral of $cos(x)$ between zero and $\pi/2$, approximated by trapezoids.

Here's how `trapint` might be used in a program:

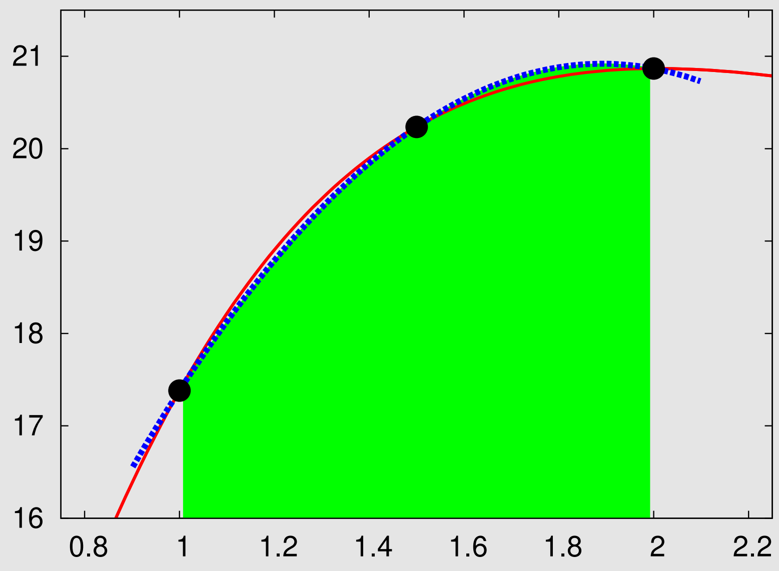
```
area = trapint( cos, 0, M_PI/2.0, 5 );
```

The line above would calculate an estimate of the area under the cosine function between zero and $\frac{\pi}{2}$ radians, using five trapezoidal slices. (See Figure 10.19.)

But what about...?

Are trapezoids the best possible shape for numerical integration? Not necessarily. There are other techniques that work better in some circumstances. Sometimes these techniques will give a more accurate estimate of the area. Sometimes they'll give you an estimate more quickly.

One common alternative to the "trapezoid rule" is called "Simpson's rule", named after 18th-Century British mathematician Thomas Simpson. Instead of connecting two data points with a straight line to make the top of each slice, Simpson's rule draws a section of a parabola through *three* adjacent points.



In the graph above, the dashed line represents a section of a parabola that approximates the shape of the curve. This parabola forms the top of slice. As you can see, Simpson's rule often fits a curve better than the trapezoid rule. The area of the slice can be determined from the parameters of this parabola and the width of the slice.

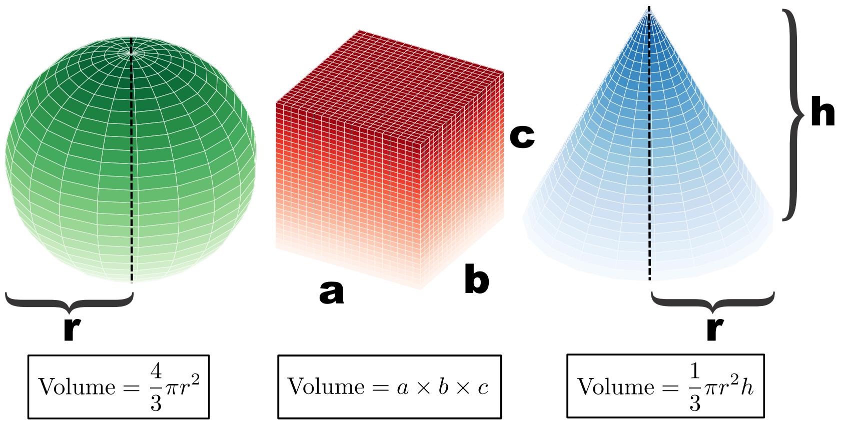
There are many other numerical integration schemes. One of them is the Monte Carlo method we'll talk about in a later section of this chapter. It doesn't use slices at all!

Thomas Simpson, 1710-1761.

Source: University of St. Andrews

10.10. Estimating Volume

The integrals we've used so far measure 2-dimensional areas. Sometimes we'll also need to estimate 3-dimensional volumes. We know nice mathematical formulas that tell us the volume of some simple geometrical shapes: a sphere, a rectangular box, or a cone, for example.


r
a
b
c
h
r
Volume $= \frac{4}{3}\pi r^2$
Volume $= a \times b \times c$
Volume $= \frac{1}{3}\pi r^2 h$

But sometimes we want to find the volume of a more complicated shape. That's OK. We can estimate the shape's volume even if we don't have a formula that will give us an exact value. One way to do this is by breaking the complicated shape up into simpler shapes, then adding up their volumes. The best way of doing this will depend on the particular shape we're dealing with, but let's look at a technique that will work with one group of common shapes.

Imagine that we're given a shape that's cylindrically symmetric about some axis, like the lovely vase in Figure 10.20. By cylindrically symmetric, I mean that the vase would look the same if we rotated it around on the table. A cylinder or a cone would also be cylindrically symmetric shapes.

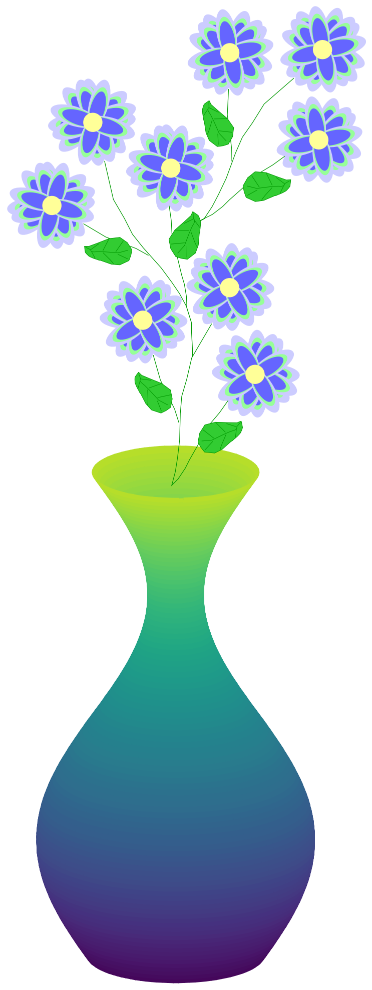


Figure 10.20: This vase has a cylindrically symmetric shape.

Because of the vase's symmetry, we could describe its shape completely by just specifying the shape of its sides. This might be easier to see if we lay the vase on its side, as in Figure 10.21. Now we can see that this is similar to the 2-dimensional problems we did earlier. The curve of the vase's side could be approximately sliced into something like trapezoids, and we could add up the volumes of these shapes to get an estimate of the vase's total volume.

We're not calculating areas here, though. Now we're calculating volumes. The slices of our vase won't be trapezoids, they'll be some 3-dimensional shape. As you can see from Figure 10.22, the slices will be truncated cones.

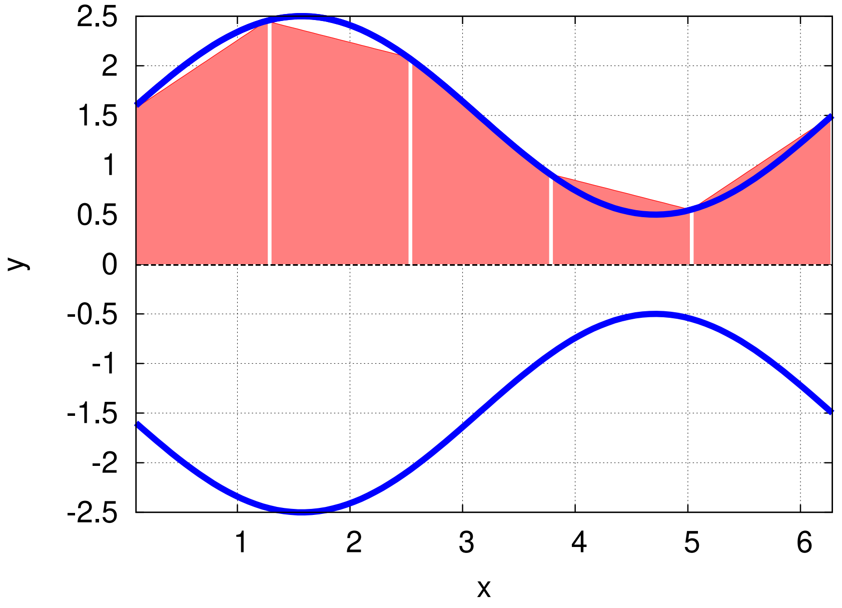


Figure 10.21: We can graph the curves that define the shape of the vase, and approximate them with simpler shapes.

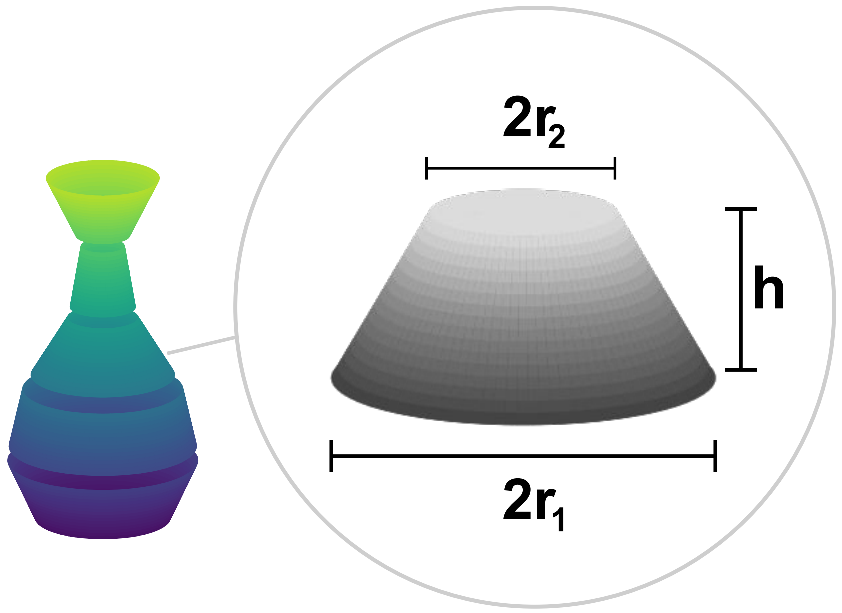


Figure 10.22: The vase's volume could be sliced into a bunch of truncated cones that approximate its shape. The diameter of the top is $2r_2$ and the diameter of the bottom is $2r_1$, where r_2 and r_1 are the radius of top and bottom.

Fortunately there's a formula that will tell us the volume of a truncated cone, given its height, h, and the radius of its top and bottom circles, r_1 and r_2. (Notice that in this case it doesn't matter which is top and which is bottom. The volume would be the same if we flipped the shape over.) The formula is:

$$\text{Volume} = \frac{1}{3}\pi(r_1^2 + r_1 r_2 + r_2^2)h$$

The values of r_1 and r_2 for each truncated cone are given by the height of the curve in Figure 10.21. For a real vase, we could get these values by just measuring the diameter of the vase at a few points. The data might look like this:

```
0.0 3.2
1.2 4.9
2.5 4.1
3.7 1.8
4.9 1.1
6.2 3.0
```

where the first column is the height above the tabletop and the second column is the vase's diameter at that height. If we put these data into a file named `vase.dat` we could write a program like Program 10.5 (`vase.cpp`) to read the file and estimate the vase's volume.

Notice that this program looks a look like our earlier `roadtrip.cpp` program (Program 10.3). It uses the same strategy for reading an unknown number of data points from a file and using the intervals between them. The main difference is that now we're calculating the volumes of truncated cones, using the formula above, whereas in the earlier program we were calculating the areas of trapezoids.

We could use this this new program to estimate the volume of any cylindrically-symmetric shape, based on some measurements of height and diameter stored in a data file.

Program 10.5: vase.cpp, Estimating volume based on measurements

```
#include <stdio.h>
#include <math.h>
int main () {
  double height, diameter;
  double oldheight, olddiameter;
  double h,r1,r2;
  double volume=0.0;
  int nmeasurements=0;
  FILE *input;

  input = fopen("vase.dat","r");
  while ( fscanf( input, "%lf %lf", &height, &diameter ) != EOF ) {
    if ( nmeasurements != 0 ) {
      r1 = olddiameter/2.0;
      r2 = diameter/2.0;
      h = height-oldheight;
      volume += M_PI*( r1*r1 + r1*r2 + r2*r2 )*h/3.0;
    }
    oldheight = height;
    olddiameter = diameter;
    nmeasurements++;
  }
  fclose ( input );

  printf ( "Total volume is %lf\n", volume );
}
```

Sometimes the volume we need to estimate won't belong to a physical object like a vase that we can take measurements from. We might be given a mathematical function that describes a shape in 3-dimensional space, and asked to find the volume inside it. This would be analogous to the 2-dimensional integration of a function that we did earlier, in Program 10.4.

Program 10.6 (`vase-func.cpp`) uses a function to describe the shape of the vase's side, then divides the vase's height up into five truncated-cone-shaped slices and adds up their volumes. It works just like Program 10.4, but calculates the volumes of truncated-cone-shaped slices instead of the areas of trapezoids. The function `func` at the top defines the shape of the vase's side, and the values `xmin` and `xmax` are the bottom and top of the vase, respectively.

Again, this program could be used to estimate the volume of any cylindrically-symmetric shape. We'd just need to change the definition of `func` and `xmin` and `xmax` appropriately.

Program 10.6: vase-func.cpp, Estimating volume based on a function

```
#include <stdio.h>
#include <math.h>
double func( double x ) {
  double value;
  value = 1.5 + sin(x);
  return(value);
}
int main () {
  double x, volume=0;
  double xmin=0.1, xmax=2.0*M_PI;
  double r1, r2, h;
  int i, nsteps=5;

  h = (xmax-xmin)/nsteps;
  x = xmin;
  for ( i=0; i<nsteps; i++ ) {
    r1 = func(x);
    r2 = func(x+h);
    volume += M_PI*( r1*r1 + r1*r2 + r2*r2 )*h/3.0;
    x += h;
  }

  printf ( "Integral from %lf to %lf is %lf\n",
           xmin, xmax, volume );
}
```

10.11. Monte Carlo Integration

Figure 10.23: Chickens, finding the approximate value of an integral.
Source: Wikimedia Commons

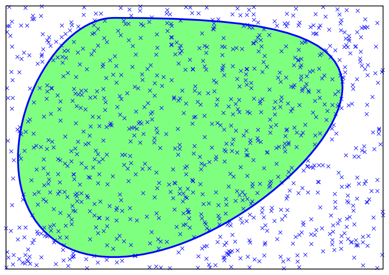


Figure 10.24: The ratio of pecks inside the shape to pecks outside the shape is approximately equal to the ratio of the shape's area to the area of its enclosure.

Did you know that chickens can do calculus? It's true. Let's say we wanted to find the area of the shaded shape in Figure 10.24. This could be any shape, possibly one whose area can't be exactly determined mathematically. But, we're smart chicken-farming programmers, so we know how to find an approximate value for the area.

We walk into our chickenyard and draw the shape on the ground, then go away and let the chickens walk about, pecking at the ground. We watch from a distance and count how many times they peck anywhere in the yard, and keep a separate count of the number of times they peck inside the shape we've drawn.

Now we're all set to estimate the area of the weird shape we've drawn. We know the dimensions of our chickenyard, and can calculate its area. We know the total number of pecks in the chickenyard, and we know how many of those pecks were inside the weird shape. To keep things clear, let's define some variables to represent these things:

A_{total}	=	The total area of the chickenyard
n_{total}	=	The total number of pecks
n_{shape}	=	The number of pecks inside the shape
A_{shape}	=	The unknown area of the shape

If the total number of pecks is large and evenly spread through the chickenyard we'd expect that the following will be true:

$$\frac{A_{shape}}{A_{total}} = \frac{n_{shape}}{n_{total}}$$

Or, rearranging a little, we could say that

$$A_{shape} = A_{total} \frac{n_{shape}}{n_{total}} \tag{10.1}$$

We know all of the things on the right-hand side of this equation, so we just need to plug them into a calculator to find A_{shape}.
Thanks Chickens!

Fortunately for those who live in apartments, we don't need chickens in order to use this technique. We can do the same thing with a computer program.

As we've seen before, we can use the `rand` function to generate random numbers. Instead of letting chickens peck, we can write a program that generates pecks at random locations. Beyond that, the only thing our program will need will be some way of knowing whether a particular point lies inside the shape we're interested in.

Let's try doing it with a shape whose area we know exactly, so we can see if our pecking technique really does do a good job of estimating the area. Figure 10.25 shows a circular area inside a square chickenyard. The circle has a radius of 1, and we know that the area of a circle is πr^2, so the area of this circle should be exactly π.

"A balmy, restful peacefulness seemed to reign everywhere. Even the old hen seemed well satisfied. She scratched among the stones and called to her chickens when she found a treasure; and all the while clucked to herself with intense inward satisfaction. Waldo, as he sat with his knees drawn up to his chin and his arms folded on them, looked at it all and smiled. An evil world, a deceitful, treacherous, mirage-like world, it might be; but a lovely world for all that, and to sit there, gloating in the sunlight, was perfect. It was worth having been a little child, and having cried and prayed, so one might sit there."
—*The Story of an African Farm*, Olive Schreiner (1883).
Source: Wikimedia Commons

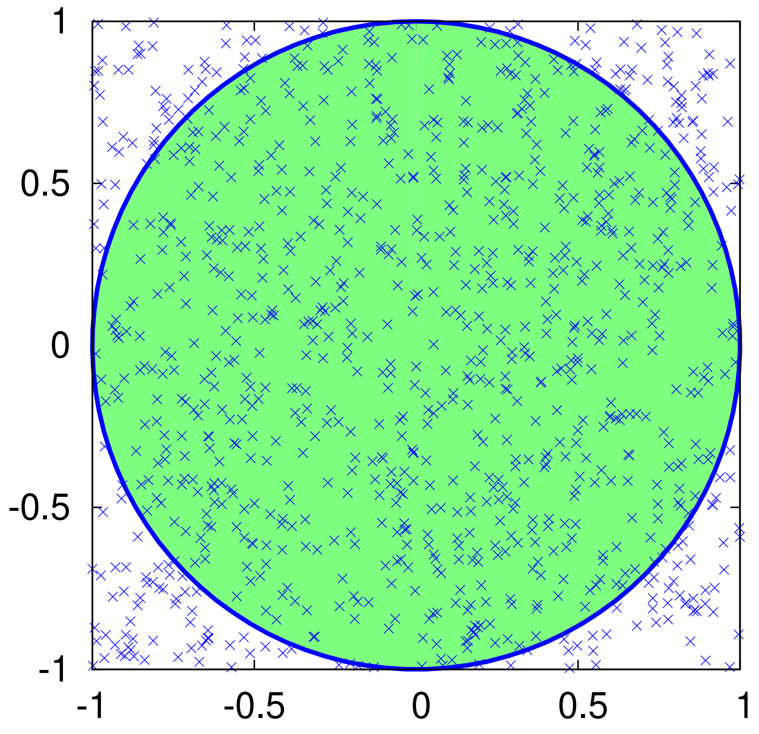


Figure 10.25: A circular area with a radius of 1, inside a square 1 × 1 "chickenyard".

Program 10.7 estimates the area of the circle by simulating chicken pecks. It generates 1,000 random peck positions inside the chicken yard, then checks each peck to see if it's inside our area of interest. The program keeps track of the number of pecks inside the area. After it's done pecking, the program calculates the area by using Equation 10.1, above.

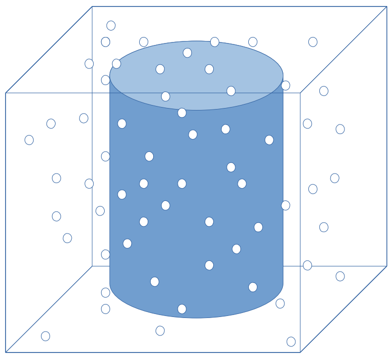


Figure 10.26: A 3-d version of our pecking scheme.

Note that each random peck position is created by generating random values for x and y. We do this by multiplying the width or height of the chickenyard by a random number between 0 and 1, and then adding the result to the minimum x or y value.

The function `inside` tells the program whether a given peck lies inside the area we're interested in. it checks to see if $\sqrt{x^2 + y^2}$ is less than or equal to the circle's radius. If the peck is inside, the function returns the value "1". Otherwise, it returns a "0".

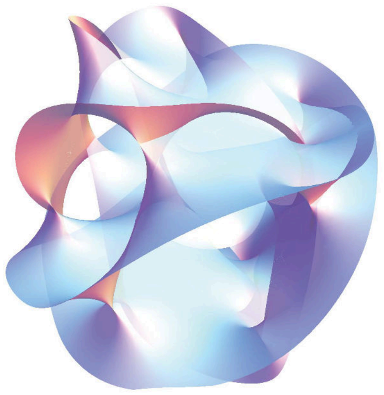
The `inside` function is the only part of the program that's specific to a circle. If we wanted to find the area of a different shape, we'd only need to rewrite this function.

The Monte Carlo Casino, in the Principality of Monaco

Source: Wikimedia Commons

The program could be extended to three dimensions by adding a z coordinate, and used to estimate volumes. In that case, we'd generate random x, y, and z coordinate for each "peck", and keep track of how many landed inside a 3-d shape we were interested in (see Figure 10.26).

The technique described in this section is known as "Monte Carlo integration". It takes its name from the gambling resort of Monte Carlo, on the French Riviera. Like the dice-rolling gamblers at Monte Carlo, our pecking program does its job by generating random numbers.


Consider the case of an 11-dimensional integral in String Theory. This might be utterly impossible to solve exactly, but by generating thousands of sets of 11 coordinate values, we could estimate its value by Monte Carlo methods.

Source: Wikimedia Commons

The Monte Carlo technique has several virtues:

- As we mentioned above, the Monte Carlo method can easily be extended to higher dimensional problems.
- To use this technique you only need two things:
  - You need a way to determine whether a point is inside the shape you're interested in, and
  - You need to be able to draw a "chickenyard" of a known area that completely encloses the shape.
- Although other methods are often more efficient for integrating 2-d functions, the Monte Carlo technique is quite efficient for higher-dimensional integrals.

Program 10.7: peck.cpp

```
#include <stdio.h>
#include <stdlib.h>
#include <time.h>
#include <math.h>

int inside ( double x, double y ) {
  double radius = 1.0;
  if ( sqrt( x*x + y*y ) <= radius ) {
    return ( 1 );
  } else {
    return ( 0 );
  }
}

int main () {
  double xmin = -1, xmax = 1;
  double ymin = -1, ymax = 1;
  double atotal;
  double ashape;
  double xrange, yrange, x, y;

  int ntotal = 1000;
  int peck;
  int nshape=0;

  xrange = xmax - xmin;
  yrange = ymax - ymin;

  srand(time(NULL));

  for ( peck=0; peck<ntotal; peck++ ) {
    x = xmin + xrange*rand()/(1.0+RAND_MAX);
    y = ymin + yrange*rand()/(1.0+RAND_MAX);

    if ( inside( x, y ) ) {
      nshape++;
    }
  }

  atotal = (xmax-xmin) * (ymax-ymin);
  ashape = atotal * nshape/ntotal;

  printf ( "The area of the shape is %lf\n", ashape );
}
```

Exercise 54: Chicken Pot Pi

Create, compile, and run Program 10.7. Does it give a good approximation of the value of π? Try increasing the number of pecks to 100,000. Does this improve the program's results?

Now try integrating the `sin(sin(x))` function again, this time using the Monte Carlo technique. To do this, make a modified version of Program 10.7 as follows:

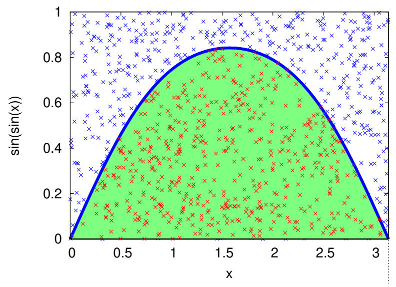


Figure 10.27: Integrating $sin(sin(x))$ using the Monte Carlo method.

- Copy `peck.cpp` to `peck2.cpp`:

```
cp peck.cpp peck2.cpp
```

- Change the `inside` function so it looks like this:

```
int inside ( double x, double y ) {
  if ( y <= sin(sin(x)) ) {
    return ( 1 );
  } else {
    return ( 0 );
  }
}
```

- Change the limits of the "chickenyard" to these values:

```
  double xmin = 0, xmax = M_PI;
  double ymin = 0, ymax = 1;
```

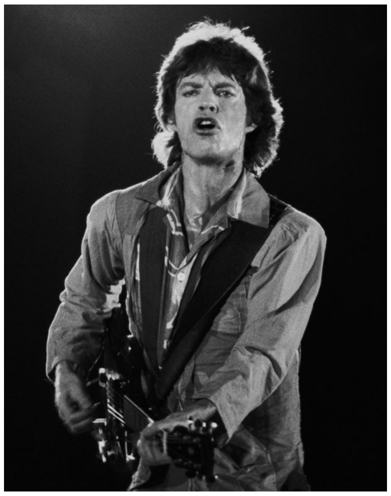
Compile and run your new `peck2.cpp` program. Does its result agree with the value you got earlier using the trapezoid rule to integrate `sin(sin(x))`?

$sin(x)dx = 2$

Figure 10.28: A chickintegral

10.12. Conclusion

The trapezoid rule and Monte Carlo integration techniques are both useful tools for dealing with recalcitrant integrals. They let a programmer find approximate values for the area under a curve defined by data points, and find arbitrarily precise approximations to a wide array of definite integrals of functions, many of which can't be solved exactly.


As Mick Jagger says, "You can't always get what you want, but if you try sometimes you just might find you get what you need."
Source: Wikimedia Commons

Practice Problems

1. Copy Program 10.4 into a new program named `cosarea.cpp` and modify it so that it estimates the integral of the cosine function between $x = 0$ and $x = \pi/4$ using ten slices. (See Figure 10.29.) The answer answer should be about 0.7.

2. Figure 10.30 shows the shape of the Rio Grande river bed near Bernalillo, New Mexico. Researchers measured the depth of the water at various positions as they waded across the river from one bank to the other. The resulting graph shows what a slice through the river would look like. We can use this data to estimate the cross-sectional area of the river. It would just be the shaded area shown in Figure 10.30, which is the area between the water's surface and the streambed. The data collected by the researchers is shown in Figure 10.31.

   Using Program 10.3 as a starting point, write a program named `riogrande.cpp` that finds the river's cross-sectional area. The program should read a data file named `rio-grande.dat` that contains the data in Figure 10.31. At the end, the program should print an estimate of the area, in square feet. (Note that the answer will be negative since all of the heights are below the water surface. You can just ignore the minus sign.)

   Your program should come up with an area of about 480 square feet. If we know this area, and we measure how fast the water is flowing (by dropping in a leaf and timing it between two points, for example) we can calculate the flow rate of water through the river. For example, if the water is flowing at 2 feet per second, that would mean that the flow rate is 480 × 2 = 960 cubic feet per second. That's over 7,000 gallons per second!

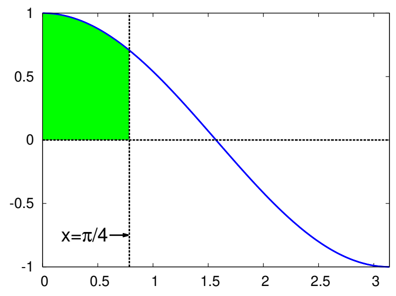


Figure 10.29: A graph of the cosine function. The region between $x = 0$ and $x = \pi/4$ is shaded.

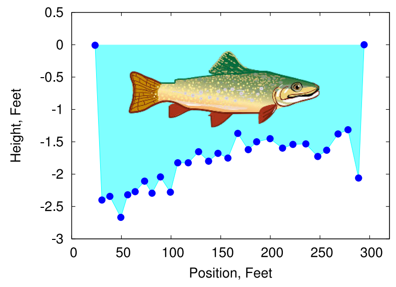


Figure 10.30: A cross-section of the Rio Grande near Bernalillo, NM. (Trout not to scale.)
Data from: J.E. Veenhuis, 2002, Summary of Flow Loss between Selected Cross Sections on the Rio Grande in and near Albuquerque, New Mexico: U.S. Geological Survey Water-Resources Investigations Report 02-4131.

```
24 -0.0
30 -2.4
38 -2.3
49 -2.7
56 -2.3
64 -2.3
73 -2.1
81 -2.3
89 -2.0
99 -2.3
107 -1.8
117 -1.8
128 -1.7
138 -1.8
147 -1.7
157 -1.8
167 -1.4
178 -1.6
186 -1.5
200 -1.5
212 -1.6
223 -1.5
236 -1.5
248 -1.7
257 -1.6
268 -1.4
278 -1.3
289 -2.1
294 0.0
```

Figure 10.31: Rio Grande cross-section data. The first column is distance and the second column is water depth, both measured in feet.

The Rio Grande near Bernalillo, New Mexico. *Source: Wikimedia Commons*

3. The Lorentzian function is often used to describe the shape of lines in a spectrum. It can be written as:

$$\frac{1}{\pi\gamma\left[1+\left(\frac{x-x_0}{\gamma}\right)^2\right]}$$

where x_0 and γ are parameters that control the shape of the function. Figure 10.32 shows a Lorentzian function with $x_0 = 0$ and $\gamma = 2$, over the range $x = -10$ to $x = 10$, divided into five or six slices.

Write a program named `lorentzian.cpp` that estimates the area under this curve. Look at Program 10.4 for inspiration. The program should ask you (using `scanf`) how many slices you want to use, and it should print the estimated area when it's done.

If you start out with a small number of slices (two, for example) and try the program several times with increasing numbers of slices, you should see that the estimates of the area jump up and down at first, then settle into a value of around 0.87. Figure 10.32 shows why this happens. For a small number of slices, odd numbers tend to underestimate the curve's area, and even numbers tend to overestimate it.

(**Note:** The symbol γ is the Greek letter *gamma*, but you'll run into trouble if you try to have a variable named `gamma` in your program. This is because C's math library contains a function named `gamma`. If you give a variable the same name, the compiler will get confused. Instead, you might use `g` to represent γ in your program.)

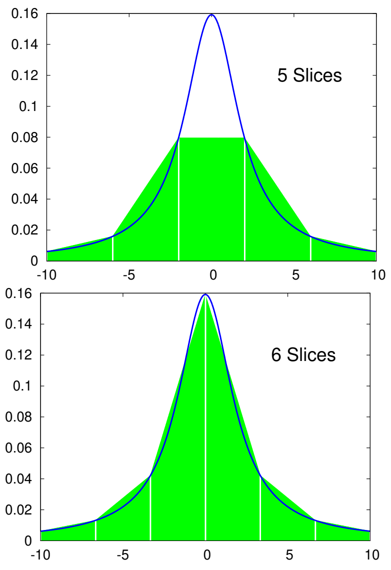


Figure 10.32: A Lorentzian function divided into five and six slices.

4. Program 10.7 (`peck.cpp`) estimates the area of a circle using the Monte Carlo method. Copy this program into a new file named `sphere-peck.cpp` and modify the new program so that it estimates the volume of a sphere.

To do this, you'll need to add a `z` dimension to go along with the `x` and `y` dimensions that are in the old program. That means adding variables named `z`, `zmin`, `zmax`, `zrange`.

The volume of the "chickenyard" will be the box surrounding the sphere, as shown in Figure 10.33. The volume of a box is just its height $\times$ width $\times$ depth.

You'll need to modify your `inside` function so that it takes three arguments instead of just two. The distance of a "peck' from the origin will be $\sqrt{x^2 + y^2 + z^2}$.

The real volume of the sphere is $\frac{4}{3}\pi$ or approxmiately 4.19. Your program should come up with a value similar to this if you use a thousand or more pecks.

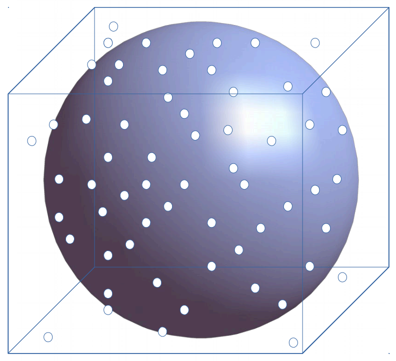


Figure 10.33: A sphere in a box. The "pecks" are represented by dots.

5. Find the approximate area under the curve $y = 1 - x^2$ between $x = -1$ and $x = 1$ (see Figure 10.34) using two different techniques:

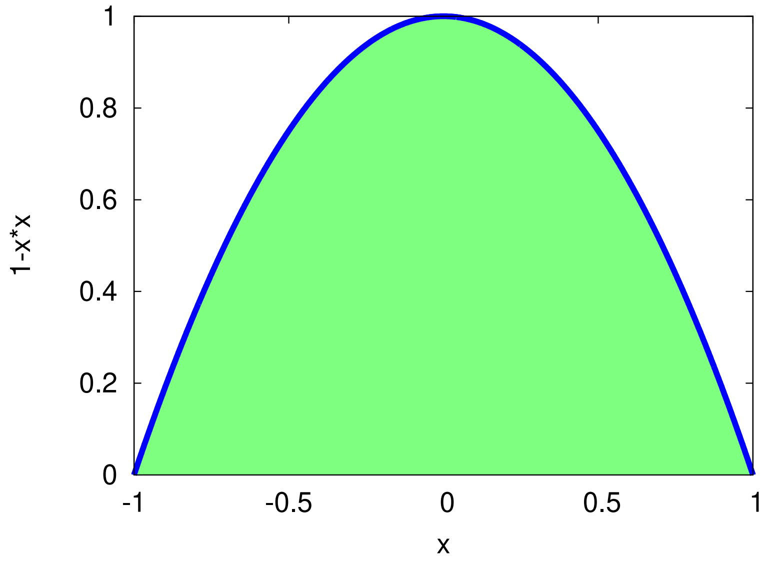


Figure 10.34: A parabolic area between $x = -1$ and $x = 1$.

(a) Copy Program 10.4 into a new file named `parabola.cpp`. Modify the new program so that, instead of finding the integral of the sine function, it finds the integral of $1 - x^2$ over the range from $x = -1$ to $x = 1$.

   To do this, you'll need to change the `func` function and you'll need to change the values of `xmin` and `xmax`.

   Compile and run your program. It should find that the area is approximately $\frac{4}{3}$. If the result you get initially isn't very close, try increasing the value of `nsteps` (the number of slices).

(b) Copy Program 10.7 into a new file named `parabola-peck.cpp`. Modify the program so that it finds the integral of $1 - x^2$ over the range from $x = -1$ to $x = 1$.

   To do this, you'll need to modify the `inside` function. The function should return a 1 for points in the shaded region of Figure 10.34 and zero otherwise. (**Hint:** Check to see if `y < 1-x*x`.) You'll also need to change the bounds of your chickenyard, since `ymin` should now be zero.

   Compile and run your program. It should find that the area is approximately $\frac{4}{3}$. If the result you get initially isn't very close, try increasing the value of `ntotal` (the number of pecks).

6. There's a set of special functions named "Bessel functions" that are used in describing the shape of a vibrating drumhead (see Figure 10.35). In C, the first Bessel function is called `j0`. Find the approximate area under the curve $y = j0(x) + 1$ between $x = 0$ and $x = 16$ (see Figure 10.36) using two different techniques, while looking to see which technique gives a good answer more quickly:

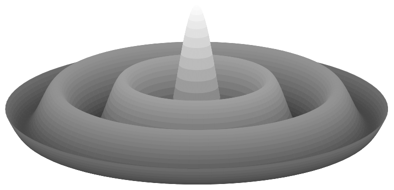


Figure 10.35: The shape of a vibrating drumhead can be described with the help of Bessel functions.

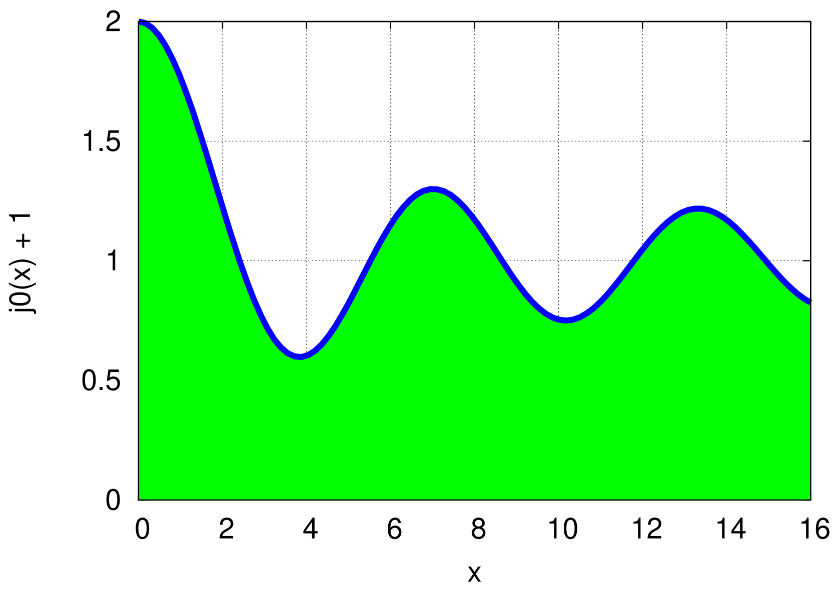


Figure 10.36: The function $j0(x) + 1$ between $x = 0$ and $x = 16$.

(a) Copy Program 10.4 into a new file named `bessel.cpp`. Modify the new program so that, instead of finding the integral of the sine function, it finds the integral of $j0(x) + 1$ over the range from $x = 0$ to $x = 16$.

Instead of setting `nsteps` equal to 5, modify the program so that it tries many different values of `nsteps`. Make a loop that tries each value of `nsteps` between 1 and 1,000. For each value, print `nsteps` and the estimated area. The result should be two columns of numbers, separated by a space.

Remember to reset your estimate of the area to zero whenever you change to a new value of `nsteps`. You'll need to modify the `func` function and you'll need to change the values of `xmin` and `xmax`.

Compile and run your program. When `nsteps` is large, you should find that the area is approximately 17.1. Notice that the program's estimate of the area is very different from this when `nsteps` is small, but it rapidly appraches the correct value.

(b) Copy Program 10.7 into a new file named `bessel-peck.cpp`. Modify the program so that it finds the integral of $j0(x) + 1$ over the range from $x = 0$ to $x = 16$.

Instead of setting `ntotal` to 1,000, try different values of `ntotal`. Make a loop that tries each value of `ntotal` between 1 and 1,000. For each value, print `ntotal` and the estimated area. The result should be two columns of numbers, separated by a space.

Remember to reset `nshape` to zero whenever you change the value of `ntotal`. You'll need to modify the `inside` function. The function should return a 1 for points in the shaded region of Figure 10.36 and zero otherwise. (**Hint:** Check to see if `y<j0(x)-1`.) You'll also need to change the bounds of your chickenyard, since `xmin` and `ymin` should now be zero, and `xmax` should be 16.

Compile and run your program. It should find that the area is approximately 17.1. Notice that the program eventually gets close to this value, but not until `ntotal` is pretty large.

If we ran our two programs like this:

```
./bessel > bessel.dat
./bessel-peck > bessel-peck.dat
```

we could compare the two output files by plotting them with *gnuplot*. The result is shown in Figure 10.37. The vertical axis shows our estimate of the area and the horizontal axis shows how many slices or pecks we needed to get that estimate. As you can see, the Monte Carlo program `bessel-peck.cpp` eventually gets close to the right answer, but it takes many "pecks" to get there. The other program (`bessel.cpp`) gets close to the right answer after only a few trapezoidal slices, and stays there from then on.

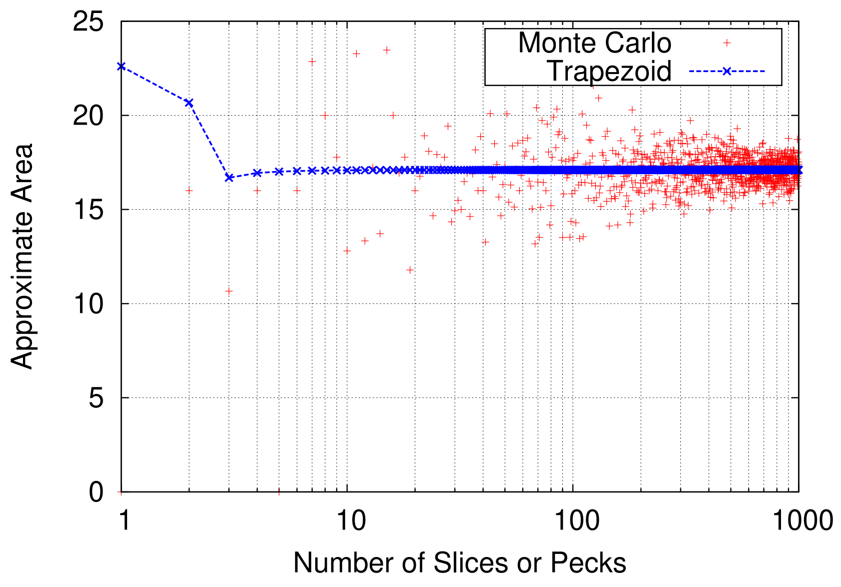


Figure 10.37: A comparison of area estimates for different values of `nsteps` or `ntotal`. Notice that the x axis is logarithmic, so we can pack a large range into it while still being able to see the small values. In *gnuplot* you can do this by saying `set log x` before you use the `plot` command.

7. Consider the graph displayed in Figure 10.38. It shows two parabolas, with the area between them shaded. The parabolas are described by these two equations:

$y = 1 - x^2$	Top parabola
$y = x^2 - 1$	Bottom parabola

   (a) Can you write a program that estimates this area using the Trapezoid Rule? Call the program `2parabola.cpp`. You might look at Program 10.4 as an example to get you started.

   (b) Can you write a program that estimates the area using Monte Carlo methods? Call this program `2parabola-peck.cpp`. Look at Program 10.7 for inspiration.

   Both programs should find that the area is about $\frac{8}{3}$ ($\simeq$ 2.67).

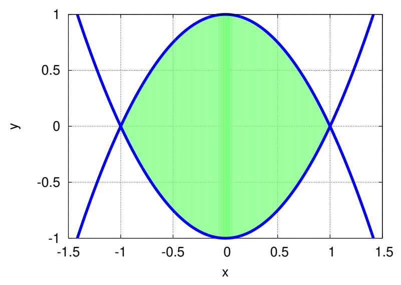


Figure 10.38: The intersection of two parabolas.